

An entropic approach to local realism and noncontextuality

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For any Bell locality scenario (or Kochen-Specker noncontextuality scenario), the joint Shannon entropies of local (or noncontextual) models define a convex cone for which the non-trivial facets are tight entropic Bell (or contextuality) inequalities. In this paper we explore this entropic approach and derive tight entropic inequalities for various scenarios. One advantage of entropic inequalities is that they easily adapt to situations like bilocality scenarios, which have additional independence requirements that are non-linear on the level of probabilities, but linear on the level of entropies. Another advantage is that, despite the nonlinearity, taking detection inefficiencies into account turns out to be very simple. When joint measurements are conducted by a single detector only, the detector efficiency for witnessing quantum contextuality can be arbitrarily low.

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I. INTRODUCTION

Quantum mechanics predicts that experiments performed by space-like separated and independent observers may display nonlocal correlations, which cannot be explained solely by past interactions. The assumption that physical quantities have well established values previous to any measurement and that signals cannot travel faster than the speed of light, as stipulated by special relativity, entails limits on the correlations the observers may obtain. Such restrictions, usually expressed as *Bell inequalities* [1], may be surpassed within quantum theory when the observers share entangled quantum states, and it is in this sense that the quantum correlations are nonlocal.

Similarly, noncontextuality is a classically well-defined property of mutually compatible observables. Two observables A and B are mutually compatible if the result for the measurement of A , even if not performed, does not depend on the prior or simultaneous measurement of B and vice versa. The notion of noncontextuality is precisely captured by the Kochen-Specker (KS) theorem [2], stating that no noncontextual hidden variable model (NCHV) can reproduce the results of quantum mechanics. Interestingly, as opposed to Bell's theorem, the KS theorem is not state-dependent and it holds for any physical system (composite or not) with a state space of dimension higher than two.

Focusing on the Bell scenario, once the particular scenario is defined—the number of spatially separated parties, the number of measurements settings for each party and the number of outcomes for each setting—the associated local (realistic) models form a convex set with a finite number of extremal points, an object known as the *local polytope* [3]. The tight Bell inequalities are the non-trivial facets bounding the local polytope. Given this geometric picture, a *nonlocal model* is a point outside the local polytope, or, equivalently, a point which violates a Bell inequality.

It is clear that to properly understand nonlocality, it should be considered from as many aspects as possible. In

the information-theoretic approach introduced by Braunstein and Caves [4], it was shown that if local realism holds, then the joint Shannon entropies carried by the measurements on two distant systems must satisfy certain inequalities, which can be regarded as *entropic Bell inequalities*. One advantage of this entropic approach is that the inequalities do not depend on the number of outcomes of the measured observables, which implies that they can readily be applied to quantum systems of arbitrary local dimension and to the consideration of detection inefficiencies [5].

In this paper, our aim is to further develop this entropic program to Bell inequalities and also introduce entropic inequalities to the study of (quantum and post-quantum) contextuality. As we discuss in Sec. II and in more detail in [6], the standard information-theoretic inequalities for joint Shannon entropies (monotonicity and submodularity) define a convex cone [7] whose projection to the joint entropies of jointly measurable observables is another convex cone, whose facets correspond to all the optimal Shannon-type entropic Bell inequalities.

From this geometrical point of view, we investigate the family of chained entropic inequalities derived in Ref. [4]—which we have shown to be the tightest entropic inequalities in the appropriate scenarios [6]—and look for violations in quantum and post-quantum probabilistic theories. In particular, we show how an entropic contextuality inequality can be violated—for joint measurements of the commuting observables—even for arbitrarily low detection efficiencies. Moreover, in Sec. III we analyze a possible connection between the entropic CHSH inequality and nonlocality distillation [8–13]. For a bilocality scenario [14] (a Bell scenario with two independent sources which allow entanglement swapping [15]), we derive in Sec. IV all the relevant tight (Shannon-type) entropic inequalities.

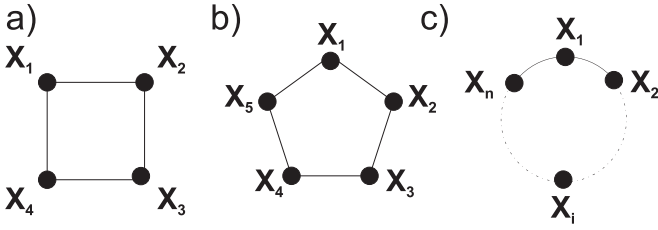


FIG. 1: (Color online) Some contextuality and nonlocality scenarios; the number of outcomes of each observable is arbitrary. **a.** CHSH scenario, 2 parties with 2 measurements settings each **b.** Klyachko scenario, 5 observables arranged in a cyclic configuration such that each observable is compatible with its neighbors **c.** Generalization of the Klyachko scenario, with n observables in a cyclic configuration (the n -cycle [16]). The unique non-trivial entropic inequality in all these cases is given by (3).

II. MARGINAL SCENARIOS AND ENTROPIC INEQUALITIES

In the following, we collect the basic definitions and results concerning entropic inequalities. For a more detailed discussion, we refer to Ref. [6].

Marginal scenarios. The concept of *marginal scenario* subsumes both Bell scenarios and contextuality scenarios. A marginal scenario is defined by specifying a set of observables X_1, \dots, X_n for which certain subsets are known to be compatible and can be jointly measured. If a subset of these observables can be jointly measured, then so can any smaller subset; therefore, the collection of subsets of jointly measurable observables should be closed under taking smaller subsets. These are the *marginal scenarios* which we have discussed in [6]; see there for more background and references on the marginal problem. The *measurement covers* of [17] capture precisely the same idea in a slightly different way.

Definition 1. A marginal scenario \mathcal{M} is a collection $\mathcal{M} = \{S_1, \dots, S_{|\mathcal{M}|}\}$ of subsets $S_i \subseteq \{X_1, \dots, X_n\}$ such that if $S \in \mathcal{M}$ and $S' \subseteq S$, then also $S' \in \mathcal{M}$.

In this sense, every Bell scenario is a marginal scenario [17, Sec. 2.4.1]: the collection of observables X_1, \dots, X_n should comprise all observables of all parties, and a subset of these observables is jointly measurable if it does not contain two different observables of the same party. For example in the bipartite case, with Alice having access to observables A_0, \dots, A_{m-1} and Bob to B_0, \dots, B_{m-1} ; if we write $X_i = A_{i-1}$ and $X_{i+m} = B_{i-1}$ for $i = 1, \dots, m$, then the set of all observables is $\{X_1, \dots, X_{2m}\}$, and the subsets of jointly measurable observables are the empty subset, the one-observable subsets, and the two-observable subsets $\{X_i, X_j\}$ where $i \leq m$ and $j > m$.

In a physical realization of a marginal scenario \mathcal{M} , one measures some joint statistics for every $S \in \mathcal{M}$. This means that one assigns a joint probability distribution to

every jointly measurable set of observables. We use notation like $P(1, 0 | X_3, X_5)$ for the probability of obtaining the outcomes $X_3 = 1$ and $X_5 = 0$ in a joint measurement of X_3 and X_5 (assuming that $\{X_3, X_5\} \in \mathcal{M}$).

If $S \in \mathcal{M}$ and $S' \subseteq S$, then one can take the marginal of the distribution assigned to S and obtain a distribution over the outcomes of the observables in S' . Naturally, this marginalized distribution should be the one assigned to S' . Requiring this property leads to the *marginal models* of [6], or, equivalently, to the *sheaf condition* and the *empirical models* of [17]. In the case of Bell scenarios, the marginal models are precisely the no-signaling boxes.

Noncontextual hidden variables. We now define when a marginal model is contextual. The most intuitive theories of physics are those where there exists a certain “hidden” variable λ , distributed according to probabilities $\varrho(\lambda) \geq 0$ with $\sum_{\lambda} \varrho(\lambda) = 1$, such that λ determines the complete future behavior of the system. Here, completeness means that the distribution $P(x | X_i, \lambda)$ of any observable X_i , given a certain value of λ , should be independent of the outcome distributions of all other observables. This implies that when X_i and X_j are jointly measurable, then their outcome distribution is given by

$$P(x_i, x_j | X_i, X_j) = \sum_{\lambda} \varrho(\lambda) P(x_i | X_i, \lambda) P(x_j | X_j, \lambda), \quad (1)$$

and similarly for cases where more than two observables are jointly measured. If there exist conditional distributions $P(x_i | X_i, \lambda)$ and a hidden variable distribution $\varrho(\lambda)$ such that (1) holds for all jointly measurable pairs $\{X_i, X_j\} \in \mathcal{M}$ and more generally for all $S \in \mathcal{M}$, then we say that we have found a *noncontextual hidden variable model*, and the given marginal model P is *noncontextual*; otherwise P is called *contextual*. In the case of Bell scenarios, the noncontextual hidden variable models are precisely the local hidden variable models, in which case we also use the standard terminology of “local” and “nonlocal”.

Following [16, Thm. 6] or [17, Thm. 8.1], we note that the noncontextuality of P is equivalent to the existence of a joint distribution

$$P(x_1, \dots, x_n | X_1, \dots, X_n) = p(x_1, \dots, x_n) \quad (2)$$

which marginalizes to the given distributions for all $S \in \mathcal{M}$.

The main question is: how is it possible to decide whether a given marginal model P in a marginal scenario \mathcal{M} is contextual or noncontextual?

Entropic inequalities. From a joint probability distribution (2), one can define the associated Shannon entropy

$$\begin{aligned} H(X_1 \dots X_n) &= - \sum_{x_1, \dots, x_n} p(x_1, \dots, x_n) \log_2 p(x_1, \dots, x_n) \end{aligned}$$

More generally, marginalizing the joint distribution to any subset $S \in \mathcal{M}$ of the observables gives a joint entropy $H(X_S)$, where we write X_S for the tuple of observables $(X_i)_{i \in S}$. This joint entropy $H(X_S)$ is also defined in any marginal model, since the distribution of X_S is known for $S \in \mathcal{M}$.

As has first been noticed in [4] and as we have developed more formally in a general framework [6], the noncontextuality of P , i.e. the existence of a joint distribution (2), implies that the $H(X_S)$, for $S \in \mathcal{M}$, satisfy certain inequalities which may be violated in some contextual models.

Definition 2. *An entropic contextuality inequality is a linear inequality in the $H(X_S)$ for $S \in \mathcal{M}$ which is satisfied whenever P is noncontextual. In the special case of a Bell scenario, we use the term entropic Bell inequality.*

If some marginal model violates a certain entropic contextuality inequality, then this inequality has witnessed the contextuality of the marginal model.

In Ref. [6], we have classified the entropic contextuality inequalities in the n -cycle marginal scenarios. This family of scenarios is defined by starting with any number $n \geq 3$ of observables X_1, \dots, X_n and assuming that X_i and X_{i+1} are pairwise jointly measurable for all $i = 1, \dots, n$, where we write $X_{n+1} = X_1$ for ease of notation. No other pairs of observables are assumed jointly measurable, and no triples of observables are assumed jointly measurable. For $n = 4$, Fig. 1a shows that this can be identified with the CHSH Bell scenario [18]. For $n = 5$ (see Fig. 1b) it is the marginal scenario considered by Klyachko [19, 20], and hence we call it the *Klyachko scenario*. For general n , it can be visualized as an n -sided polygon (Fig. 1c). Our result in [6] is that the inequalities derived in [4] are a complete set of tight entropic inequalities in these scenarios. (An entropic inequality is tight when no other entropic inequality can be strictly better than this one, so that a complete set of tight entropic inequalities completely characterizes the region of noncontextual marginal models in entropy space.) Stated more formally:

Theorem 3 ([6]). *A marginal model in this scenario is entropically noncontextual if and only if the entropic inequality*

$$H(X_i X_{i+1}) + \sum_{j \neq i, i+1} H(X_j) \leq \sum_{j \neq i} H(X_j X_{j+1}) \quad (3)$$

holds for all $i = 1, \dots, n$.

In principle, one may also want to consider inequalities containing derived entropic quantities like conditional entropies and mutual information. However, since these derived quantities are themselves nothing but linear combinations of joint entropies, every entropic inequality containing the former can be rewritten in terms of the latter. In fact, as there are no linear relations between joint entropies, every linear entropic inequality turns into

a unique normal form when expressed in terms of joint entropies [21, Sec. 13.2].

Besides this, the relevance of our general framework [6] lies in the fact that the standard computational geometry methods like Fourier-Motzkin elimination, which have been extensively used to characterize tight Bell inequalities in probability space, can also be applied to derive tight (Shannon-type) entropic contextuality inequalities and tight (Shannon-type) entropic Bell inequalities. Unfortunately, these computations are very demanding: applying the methods described in [6] to the tripartite Bell scenario with two observables per party, we have not been able to fully characterize the tight (Shannon-type) entropic Bell inequalities in this scenario.

In the following, we analyze the case $n = 4$ (the CHSH scenario) and the case $n = 5$ (the Klyachko scenario) in some more detail and analyze in particular their violation by marginal models arising from quantum theory.

A. The CHSH scenario

In the usual CHSH scenario [18], there are two distant parties A and B , each measuring one of two observables A_0, A_1 and B_0, B_1 , respectively. Each of these observables is taken to have two possible outcomes, so that the set of outcomes can be taken to be $\{-1, +1\}$. As usual for Bell scenarios, we take observables to be jointly measurable when they belong to different parties. In our framework, this can be described by the marginal scenario where the collection of jointly measurable sets of observables is given by

$$\begin{aligned} &\emptyset, \{A_0\}, \{A_1\}, \{B_0\}, \{B_1\}, \\ &\{A_0, B_0\}, \{A_0, B_1\}, \{A_1, B_0\}, \{A_1, B_1\}. \end{aligned}$$

This is illustrated in Fig. 1a. The CHSH inequality [18]

$$\begin{aligned} &CHSH \\ &= \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 \end{aligned} \quad (4)$$

together with its equivalent variants is a necessary and sufficient condition for noncontextuality (i.e. Bell locality) in this scenario.

By taking $n = 4$ and renaming the observables occurring in (3), we obtain the entropic inequality [4]

$$\begin{aligned} &H(A_1 B_1) + H(A_0) + H(B_0) \\ &\leq H(A_0 B_0) + H(A_0 B_1) + H(A_1 B_0). \end{aligned} \quad (5)$$

In order to emphasize the similarity with (4), we can rewrite this in terms of mutual information as

$$\begin{aligned} &I(A_0 : B_0) + I(A_0 : B_1) + I(A_1 : B_0) - I(A_1 : B_1) \\ &\quad - H(A_0) - H(B_0) \leq 0. \end{aligned}$$

The last two terms in the left-hand side are analogous to the classical bound of 2 in (4). We abbreviate the

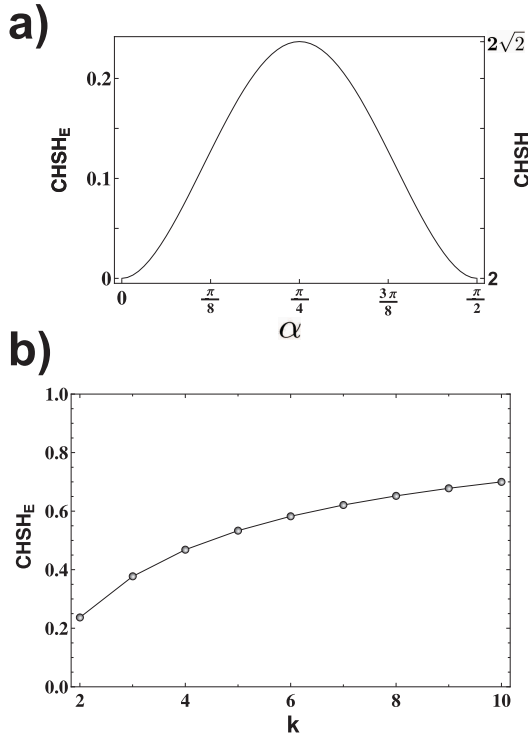


FIG. 2: (Color online) **a.** Violation profile of the entropic CHSH inequality for a pure state of the form $\cos\alpha|00\rangle + \sin\alpha|11\rangle$. The violation profile essentially coincides with the one of the standard CHSH inequality, although the scales are different. The curve is obtained by optimizing over the observables for each α . **b.** Maximal violation of the entropic inequality (3), achieved on a maximally entangled state of two qubits ($\alpha = \pi/4$) and $k = 2, \dots, 10$ measurement settings for each party.

left-hand side by $CHSH_E$, so that the *entropic CHSH inequality* is

$$CHSH_E \leq 0.$$

Note that the entropic CHSH inequality does not only apply to ± 1 -valued observables A_x and B_y ; it is completely irrelevant how many outcomes each observable has—as long as the Shannon entropies converge—and which outcomes these are.

Quantum violations of $CHSH_E$. We now assume that the two parties A and B share a pure entangled two-qubit state. For any such state, one can choose the local bases such that it has the form

$$|\psi\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle, \quad (6)$$

for some $\alpha \in (0, \pi/2)$. Writing A_x and B_y also for the ± 1 -valued quantum observables measured on $|\psi\rangle$, we obtain the joint distributions, written in terms of the standard

notation of conditional probabilities,

$$P(a, b|x, y) = \left\langle \psi \left| \left(\frac{1 + (-1)^a A_x}{2} \otimes \frac{1 + (-1)^b B_y}{2} \right) \right| \psi \right\rangle. \quad (7)$$

For ease of later notation, we stipulate that the observables A_x and B_y are ± 1 -valued, but the outcomes a, b are $\{0, 1\}$ -valued.

As shown in [4], quantum correlations of the form (7) do indeed lead to violations of (5). These violations witness the non-existence of local hidden variable models for (7). Numerical optimization shows that the maximal violation of (5) for quantum correlations of the above form is achieved when all measurement settings lie in the Y - Z plane of the Bloch sphere, that is, for measurement operators A_x and B_y of the form $\sin\theta \cdot \sigma_y + \cos\theta \cdot \sigma_z$. (One could as well take them all to lie in the X - Z plane; what is important is that they lie in the *same* plane.) The maximal violation is obtained for the maximally entangled state $\alpha = \pi/4$, on which one gets $CHSH_E \approx +0.237$.

For other values of α , the maximal violation of (5), when optimized over the measurements, follows the exact same profile as for the standard correlator inequality (4) as can be seen in Fig. 2a. However, the measurements that maximize the violation of $CHSH$ are not the ones which give the maximal violation of $CHSH_E$. In fact, we will see below that those choices of observables which produce the maximal $CHSH$ value for a certain α do not violate (5). In general, for the standard CHSH scenario, the violation of the standard inequality (4) is a necessary but not sufficient condition for the violation of (5).

We have also considered the inequalities (3) for any even $n = 2k$ as entropic Bell inequalities as follows. When A chooses between ± 1 -valued observables A_1, \dots, A_k and B among B_1, \dots, B_k , then (3) becomes applicable upon taking $X_{2i} = B_i$ and $X_{2i-1} = A_i$. Fig. 2b shows our numerical results for the maximal violation of (3) on a two-qubit state (6).

No-signaling violations of $CHSH_E$. In the CHSH scenario, there is a special class of marginal models known as *isotropic boxes*, which we would now like to study. To begin, the *Popescu-Rohrlich box* (PR box) [22] is defined to be the marginal model

$$P^{\text{PR}}(a, b|x, y) = \frac{1}{4} \left[1 + (-1)^{a \oplus b \oplus xy} \right], \quad (8)$$

It is the unique marginal model which maximally violates (4). Similarly, the *isotropic box* with parameter $C \in [0, 1]$ is defined to be the marginal model

$$P^{\text{iso}}(a, b|x, y) = \frac{1}{4} \left[1 + C (-1)^{a \oplus b \oplus xy} \right]. \quad (9)$$

It corresponds to a probabilistic mixture of P^{PR} with weight C and uniform white noise P^w with weight $1 - C$.

Equivalently, an isotropic box can be characterized by having uniformly random marginals, that is

$$\langle A_x \rangle = \langle B_y \rangle = 0,$$

together with

$$\langle A_0 B_0 \rangle = \langle A_0 B_1 \rangle = \langle A_1 B_0 \rangle = -\langle A_1 B_1 \rangle \geq 0.$$

The parameter C is determined from this by $C = \langle A_0 B_0 \rangle$. Its relation to the CHSH value of the box is simply given by $CHSH(P^{\text{iso}}) = 4C$.

Any marginal model in the CHSH scenario can be transformed into an isotropic box through a local depolarization process, keeping the CHSH value (4) invariant [23]. Therefore, for many purposes it is enough to consider isotropic boxes only.

Interestingly, no isotropic box violates the entropic CHSH inequality. In particular, this applies to the PR-box, although it maximally violates the standard CHSH inequality. One can understand this by noting that entropy only probes the probability values occurring in a distribution, but not which probability values get assigned to which outcomes. This means that the PR-box is, as far as the entropies are concerned, equivalent to the marginal model P^c describing classical correlations,

$$P^c(a, b|x, y) = \frac{1}{4} \left[1 + (-1)^{a \oplus b} \right]. \quad (10)$$

Entropic quantities cannot distinguish between the perfect anti-correlation of A_1 and B_1 as it appears in P^{PR} , and the perfect correlation of A_1 and B_1 as in P^c . All joint entropies of the PR-box coincide with those of P^c .

For two-outcome measurements, the maximal violation of (5) is +1 for the following reason: any marginal model with two-outcome measurements will satisfy $H(A_0) \leq H(A_0 B_0)$ and $H(B_0) \leq H(A_1 B_0)$; similarly, $H(A_1 B_1) \leq 1 + H(B_1) \leq H(A_0 B_1) + 1$. Taking these inequalities together shows that $CHSH_E \leq 1$ for any such marginal model in the CHSH scenario. This bound on the violation can indeed be achieved by the no-signaling box

$$P^{\text{max}} = \frac{1}{2} P^{\text{PR}} + \frac{1}{2} P^c,$$

which is an equal mixture of the PR-box with classical correlations. P^{max} can be understood as the probabilistic model in which each of the three pairs (A_0, B_0) , (A_0, B_1) and (A_1, B_0) displays perfect correlation, while the fourth pair (A_1, B_1) is uncorrelated; see also [6, Prop. 4.3]. Note that P^{max} achieves a value of 3 on CHSH, and therefore does not have a quantum-mechanical realization since it is beyond Tsirelson's bound of $2\sqrt{2}$.

This example shows that a convex combination of two non-violating marginal models may violate an entropic contextuality inequality. This highlights the strongly non-linear character of entropic inequalities (see also Fig. 4).

Discussion. It is a basic feature of Shannon entropy that the entropy of a probability distribution is invariant under permutations of the sample space. From this point of view, we find it surprising that the entropic inequality (5) can be violated at all. Violations of entropic contextuality inequalities witness a very particular kind

of contextuality: if a probabilistic model violates an entropic inequality, then so does every other probabilistic model obtained by permuting the outcome probabilities of a joint measurement, provided that the permuted joint distribution has the same marginals. For example, this leads to the phenomenon observed above that the PR-box P^{PR} is entropically indistinguishable from classical correlation P^c .

Along similar lines, the only symmetry operations that can be applied in order to transform an entropic contextuality inequality into an equivalent one are permutations of the observables which map jointly measurable sets to jointly measurable sets. In the case of a Bell scenario, these permutations are either permutations of the parties, permutations of the observables of some party, or arbitrary combinations thereof. Relabelings of the outcomes of an observable do not change the inequality, again due to the fact that entropies are invariant under outcome permutations.

B. The Klyachko scenario

A very simple state-dependent proof of the Kochen-Specker theorem with only five two-outcome observables was given by Klyachko et al. in [19, 20]. The marginal scenario in this case is the one depicted in figure 1b: there are five ± 1 -valued observables X_1, X_2, X_3, X_4, X_5 such that X_i and X_{i+1} (modulo 5) are compatible.

The so-called Klyachko inequality is a necessary and sufficient condition for noncontextuality in this scenario [19, 20]. It is given by

$$K_5 = \sum_{i=1}^5 \langle X_i X_{i+1} \rangle \geq -3 \quad (11)$$

Due to Theorem 3, the only non-trivial entropic inequality in the Klyachko scenario is given by, up to cyclic permutations of the observables, the *entropic Klyachko inequality*

$$H(X_1 X_5) + H(X_2) + H(X_3) + H(X_4) - H(X_1 X_2) - H(X_2 X_3) - H(X_3 X_4) - H(X_4 X_5) \leq 0 \quad (12)$$

To investigate the quantum violations of the entropic Klyachko inequality, we choose two-outcome observables on \mathbb{C}^3 of the form

$$X_i = 2|v_i\rangle\langle v_i| - \mathbb{1} \quad (13)$$

with the vectors $|v_i\rangle$ given by

$$\begin{aligned} |v_1\rangle &= (0, 0, 1) \\ |v_2\rangle &= (\sin \theta, \cos \theta, 0) \\ |v_3\rangle &= \mathcal{N}^{-1} (\cos \theta \sin \phi, -\sin \theta \sin \phi, \sin \theta \cos \phi) \\ |v_4\rangle &= (0, \cos \phi, \sin \phi) \\ |v_5\rangle &= (1, 0, 0), \end{aligned}$$

	(0, 1)	(1, 0)	(0, 0)
X_1, X_5	$4\phi^2$	$4\phi^2$	$1 - 8\phi^2$
X_1, X_2	$1 - 5\phi^2$	$4\phi^2$	ϕ^2
X_2, X_3	$\frac{9}{2}\phi^2$	$1 - 5\phi^2$	$\frac{1}{2}\phi^2$

TABLE I: Joint outcome probabilities for the analytic violation. All values are up to $O(\phi^3)$.

where the normalization factor is $\mathcal{N} = \sqrt{\sin^2 \theta + \cos^2 \theta \sin^2 \phi}$. Up to choice of basis and multiplying the $|v_i\rangle$ by irrelevant phases, every configuration of 5 unit vectors $|v_1\rangle, \dots, |v_5\rangle \in \mathbb{R}^3$ with $|v_i\rangle$ orthogonal to $|v_{i+1}\rangle$ is of this form.

Since each $|v_i\rangle$ is orthogonal to $|v_{i+1}\rangle$, the observable X_i commutes with X_{i+1} , so that these two observables are compatible and we can talk about their joint measurement. Also thanks to orthogonality, their joint outcome $(X_i = 1, X_{i+1} = 1)$ never occurs.

We write $|v_i \times v_{i+1}\rangle$ for a unit vector orthogonal to both $|v_i\rangle$ and $|v_{i+1}\rangle$.

Upon measuring these observables on some initial state $|\psi\rangle \in \mathbb{C}^3$, the non-vanishing joint outcome probabilities are given by

$$\begin{aligned} P(0, 1|X_i, X_{i+1}) &= P(1|X_{i+1}) = |\langle v_{i+1}|\psi\rangle|^2 \\ P(1, 0|X_i, X_{i+1}) &= P(1|X_i) = |\langle v_i|\psi\rangle|^2 \\ P(0, 0|X_i, X_{i+1}) &= 1 - P(1|X_i) - P(1|X_{i+1}) \\ &= |\langle v_i \times v_{i+1}|\psi\rangle|^2. \end{aligned} \quad (14)$$

Numerical results. Numerical calculations show that the maximal qutrit violation of (12) with observables X_i occurs on a qutrit state of the form

$$|\psi\rangle = \frac{1}{\sqrt{1 + \sin^2 \alpha}} (\sin \alpha, \cos \alpha, \sin \alpha) \quad (15)$$

with $\alpha \approx 0.29736$ and $\theta = \phi \approx 0.24131$, for which the left-hand side of (12) is $\approx +0.091$.

Analytical proof of quantum violations. We would like to present an analytic proof showing that quantum violations of (12) occur with the X_i and $|\psi\rangle$ of the above form. We set $\theta = \phi$ and $\alpha = 2\phi$ and expand everything for $\phi \ll 1$. Then, by symmetry, the joint outcome distribution of X_4 and X_5 coincides with the one of X_2 and X_1 ; likewise, the joint outcome distribution of X_3 and X_4 coincides with the one of X_3 and X_2 ; the remaining joint distributions, up to $O(\phi^3)$, are listed in Table I.

With this symmetry, (12) is equivalent to

$$\begin{aligned} H(X_1 X_5) + 2H(X_2) + H(X_3) \\ - 2H(X_1 X_2) - 2H(X_2 X_3) \leq 0. \end{aligned} \quad (16)$$

The corresponding relevant entropy values are given

by

$$\begin{aligned} H(X_1) &= -4\phi^2 \log(\phi^2) + O(\phi^2) \\ H(X_2) &= -5\phi^2 \log(\phi^2) + O(\phi^2) \\ H(X_3) &= -\frac{9}{2}\phi^2 \log(\phi^2) + O(\phi^2) \\ H(X_1 X_2) &= -5\phi^2 \log(\phi^2) + O(\phi^2) \\ H(X_2 X_3) &= -5\phi^2 \log(\phi^2) + O(\phi^2) \\ H(X_1 X_5) &= -8\phi^2 \log(\phi^2) + O(\phi^2) \end{aligned}$$

With this, the left-hand side of (16) is $-\frac{5}{2}\phi^2 \log(\phi^2) + O(\phi^2)$, which is positive for small enough ϕ .

Detection inefficiencies: single-detector model. One can take advantage of the fact that entropic inequalities can handle any finite number of outcomes and use the same approach to investigate the more realistic case with detection inefficiencies. We will consider two possible scenarios: one where compatible observables are measured jointly (one detector), and one where compatible observables are measured sequentially (two detectors).

In the single-detector model with detection efficiency $\eta \in [0, 1]$, there is an additional outcome (\emptyset, \emptyset) which represents the no-click event of the detector for each jointly measurable pair (X_i, X_{i+1}) . The new outcome probabilities P^η are given by

$$\begin{aligned} P^\eta(x_i, x_{i+1}|X_i, X_{i+1}) &= \eta P(x_i, x_{i+1}|X_i, X_{i+1}) \\ P^\eta(\emptyset, \emptyset|X_i, X_{i+1}) &= 1 - \eta, \end{aligned} \quad (17)$$

where the measurements are now $\{-1, +1, \emptyset\}$ -valued, and $x_i, x_{i+1} \in \{-1, +1\}$ are the “proper” outcomes. In this model, a no-detection event always occurs for both observables simultaneously.

The joint probabilities (17) marginalize to the single-observable distributions

$$\begin{aligned} P(x_i|X_i) &= \eta P(x_i|X_i) \\ P(\emptyset|X_i) &= 1 - \eta. \end{aligned} \quad (18)$$

Proposition 4. *With this model of inefficiencies,*

$$\begin{aligned} H^\eta(X_i) &= \eta H(X_i) + h(\eta) \\ H^\eta(X_i X_{i+1}) &= \eta H(X_i X_{i+1}) + h(\eta). \end{aligned}$$

where $h(\eta) = -\eta \log \eta - (1 - \eta) \log(1 - \eta)$ is the binary entropy.

Proof. This follows from an application of the grouping rule of Shannon entropy, see e.g. [24, Sec. 2.179]. \square

Upon plugging these equations into (12), one finds that the contributions of $h(\eta)$ cancel, so that the left-hand side simply scales as a linear function of η . Therefore, the entropic contextuality inequality (12) has violations for any $\eta \geq 0$. Moreover, the maximal violation with qutrit measurements of the form (12), (14) is given by the same state and vectors $|v_i\rangle$ which maximizes the violation for $\eta = 0$ (which are $\alpha \approx 0.29736$ and $\theta = \phi \approx 0.24131$).

Detection inefficiencies: two-detector model. We now assume that the joint measurement of X_i and X_{i+1} is realized by one detector measuring X_i and another detector measuring X_{i+1} . Again, we take each detector to have an efficiency of $\eta \in [0, 1]$, for simplicity the same value for all 5 detectors, such that the no-click event of the first detector is independent of the no-click event of the second detector. A physical situation leading to this kind of model may be a sequential scheme where the system passes through a non-demolition measurement in the first detector before reaching the second detector.

Consequently, the jointly measurable pair (X_i, X_{i+1}) has an outcome distribution, with $x_i \in \{-1, +1\}$,

$$\begin{aligned} P^\eta(x_i, x_{i+1}|X_i, X_{i+1}) &= \eta^2 P(x_i, x_{i+1}|X_i, X_{i+1}) \\ P^\eta(x_i, \emptyset|X_i, X_{i+1}) &= (1 - \eta) \eta P(x_i|X_i) \\ P^\eta(\emptyset, x_{i+1}|X_i, X_{i+1}) &= (1 - \eta) \eta P(x_{i+1}|X_{i+1}) \\ P^\eta(\emptyset, \emptyset|X_i, X_{i+1}) &= (1 - \eta)^2 \end{aligned} \quad (19)$$

which again marginalize to the single-observable distributions (18).

Proposition 5. *With this model of inefficiencies,*

$$\begin{aligned} H^\eta(X_i) &= \eta H(X_i) + h(\eta) \\ H^\eta(X_i X_{i+1}) &= \eta^2 H(X_i X_{i+1}) \\ &\quad + \eta(1 - \eta) [H(X_i) + H(X_{i+1})] + h(\eta). \end{aligned} \quad (20)$$

where $h(\eta) = -\eta \log \eta - (1 - \eta) \log(1 - \eta)$ is the binary entropy.

Proof. Again, this follows from an application of the grouping rule of Shannon entropy [24, Sec. 2.179]. \square

Due to the additional terms in (20), the required detection efficiency for witnessing quantum violations in the two-detector model turns out to be very high, $\eta \approx 0.995$.

III. NONLOCALITY DISTILLATION AND THE ENTROPIC CHSH INEQUALITY

As we have seen in the previous sections, finding violations of an entropic contextuality inequality, for example of the entropic CHSH inequality (5), is not easy, and there are few quantum-mechanical models which do violate them. Therefore, we regard the violation of an entropic contextuality inequality (entropic Bell inequality) as a witness of a very strong form of contextuality (nonlocality). So what does the violation of an entropic contextuality inequality tell us about the violating model? Can the violation be regarded as a resource for something?

Since Shannon entropy is an asymptotic quantity which measures the effective size of a probability distribution on the level of many copies (see for example the Asymptotic Equipartition Property [21]), we also expect any answers to these questions to be concerned with the limit of many copies.

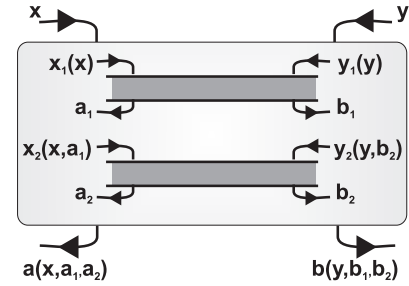


FIG. 3: (Color online) General nonlocality distillation protocol with two boxes. The inputs for the second box may locally depend on the output of the first one and the final output of the combined boxes is a function of all inputs and outputs.

In this section, we would like to consider the case of the CHSH scenario and investigate a bit, on a purely phenomenological level, one particular property which also has an asymptotic flavor: the property of *nonlocality distillation*. However, we barely have a precise hypothesis to offer—let alone a proof—and the similarities we will describe in the following may easily turn out to be superficial.

Considering the CHSH scenario, it was recently shown that nonlocality can be distilled [9–13]: by locally processing several copies of certain bipartite no-signaling boxes, one can increase the amount of nonlocality according to a particular nonlocality measure. In the following, we consider bipartite no-signaling boxes with binary inputs and outputs. Each party can wire boxes together using classical circuitry to produce a new binary-input/binary-output box.

The general distillation protocol with two copies of a no-signaling box is displayed in Fig. 3. A certain protocol distills the nonlocality if the no-signaling box obtained after the wiring is more nonlocal—according to a certain measure—than the original box. As a measure of nonlocality, we will consider the *nonlocal part* of the *EPR2 decomposition* [25], defined as follows. Any no-signaling box P can be decomposed into convex combinations of a purely local box P^L and another box P^{NL} ,

$$P = (1 - q)P^L + qP^{NL}, \quad (21)$$

with some coefficient $q \in [0, 1]$. The minimal coefficient q , when minimized over all such decompositions is the *nonlocal content* of P . By definition, computing the nonlocal content is a linear program. In the case of the CHSH scenario, there is a linear relationship between the nonlocal content and the violation of the standard CHSH inequality [13]; therefore, in the present context we may regard the *CHSH* value as our measure of nonlocality.

We now focus on two particular wirings. The first one was proposed in [9] and is given by

$$\begin{cases} x_1 = x, x_2 = x, a = a_1 \oplus a_2 \\ y_1 = y, y_2 = y, b = b_1 \oplus b_2, \end{cases} \quad (22)$$

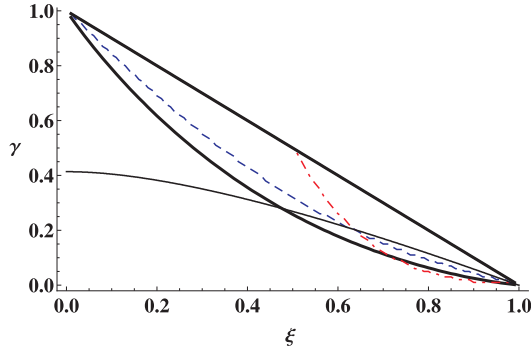


FIG. 4: (Color online) Slice of the no-signaling polytope corresponding to the family of boxes (25). The quantum region and $CHSH_E$ violations are bounded respectively by the thin black and thick black curves. The regions of distillability by the wirings (22) and (23) correspond respectively to the dashed-dotted red and dashed blue lines. The region of violations of the entropic CHSH inequality is strictly larger than the region of correlations distillable with the wiring (23). On the other hand, in a small part of the quantum region, nonlocality can be distilled by the protocol (22) while not being detected by $CHSH_E$.

while the second one was proposed in [13] and is given by

$$\begin{cases} x_1 = x, x_2 = x \oplus a_1 \oplus 1, a = a_1 \oplus a_2 \oplus 1 \\ y_1 = 1, y_2 = yb_1, b = b_1 \oplus b_2 \oplus 1. \end{cases} \quad (23)$$

As shown in [13], these two wirings completely characterize the distillability in the CHSH scenario on the two-copy level.

For example, for the wiring (22), the wired no-signaling box is given by

$$P'(a, b|x, y) = \sum_{\substack{a_1 + a_2 = a \\ b_1 + b_2 = b}} P(a_1, b_1|x, y) P(a_2, b_2|x, y), \quad (24)$$

As for any wiring protocol with two copies of the original box, the new box P' is a quadratic function of the original one. So the first (superficial) similarity between wirings and the entropic CHSH inequality is nonlinearity.

Interestingly, in the case of isotropic boxes—which we found in II A not to violate the entropic CHSH inequality—distillation is not possible [8]. More generally, we have considered the family of boxes

$$P_{\gamma, \xi} = \gamma P^{PR} + \xi P^c + (1 - \gamma - \xi) P^f, \quad (25)$$

with parameters $\gamma, \xi \in [0, 1]$, P^{PR} and P^c given respectively by (8) and (10), as well as

$$P^f(a, b|x, y) = \frac{1}{8} \left[2 + (-1)^{a \oplus b \oplus xy} \right], \quad (26)$$

which is $P^f = \frac{1}{2}P^{PR} + \frac{1}{2}P^w$, half-way between the PR-box and white noise.

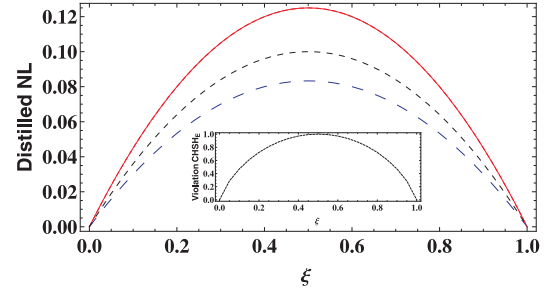


FIG. 5: (Color online) Distilled nonlocality (increase in nonlocal content) of the family (27) upon application of the wiring (30) for $d = 2, 4$ (red full line), $d = 3$ (blue large dashed line) and $d = 5$ (black dashed line) as computed by linear programming. The inset shows the violation of $CHSH_E$, which does not depend on d .

For $\gamma = 0$, this is a family of boxes with $CHSH = 0$. The family (25) forms a triangle in the no-signaling polytope extending from the boundary of the local polytope up to the PR-box. As displayed in Fig. 4, the subset of boxes violating $CHSH_E$ and the subset of boxes distillable via (23) or (22) are very similar. We expect that by going beyond the two-copy level and considering all possible wiring protocols, the subset of distillable boxes will enlarge. It seems reasonable to ask whether it will contain all boxes violating $CHSH_E$. In general, could the violation of $CHSH_E$ be a sufficient condition for distillability?

Moreover, we have investigated the bipartite scenario with two d -outcome observables per party, to which $CHSH_E$ still applies. We have considered the family of no-signaling boxes

$$P_\xi = \xi P_d^{PR} + (1 - \xi) P_d^c, \quad (27)$$

where P_d^{PR} is the generalized PR-box [26]

$$P_d^{PR}(a, b|x, y) = \begin{cases} 1/d & \text{if } a - b \equiv xy \pmod{d} \\ 0 & \text{otherwise} \end{cases}, \quad (28)$$

while P_d^c is the classically correlated box

$$P_d^c(a, b|x, y) = \begin{cases} 1/d & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}. \quad (29)$$

It follows from the definition (29) and an application of the CGLMP inequality [27] that the nonlocal content of P_ξ is simply ξ . On the other hand, $CHSH_E = -\xi \log \xi - (1 - \xi) \log 1 - \xi$ turns out to be the binary entropy.

To probe the nonlocality distillation of P_ξ , we have considered the wiring

$$\begin{cases} x_1 = x, x_2 = xa_1 \bmod 2, a = (a_1 + a_2) \bmod d \\ y_1 = y, y_2 = yb_1 \bmod 2, b = (b_1 + b_2) \bmod d \end{cases}, \quad (30)$$

which can be regarded as a generalization of the wiring proposed in Ref. [10]. Fig. 5 shows the increase of nonlocal content achievable with this protocol for any $d = 2, \dots, 5$ and the value of $CHSH_E$; we observe qualitatively identical behavior.

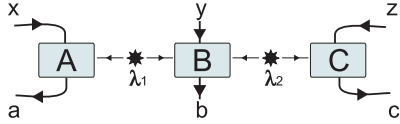


FIG. 6: (Color online) Swapping experiment scenario, where two independent entangled states are prepared and sent to three distant parts. If the central part B measures in an entangled basis, the subsystems A and C, that had no previous contact, become nonlocally correlated.

IV. ENTROPIC BILOCALITY INEQUALITIES

Bilocality scenarios. In entanglement swapping [15], there are two sources of entangled quantum states. The first source (on the left in Fig. 6) sends half of its entangled state to A and the other half to B; the second source (on the right in Fig. 6) sends half to B and half to C. If B applies the right kind of entangled measurement between the two quantum systems which he receives, then, conditioned on an outcome of this measurement, the post-measurement state between A and C will be entangled.

This idea has been used to obtain strong bounds [14] on the existence of local hidden variable models of entangled quantum states augmented by the assumption that the hidden variables λ_1 and λ_2 describing the two sources are probabilistically independent; this is the *bilocality* assumption.

Under the assumption of local realism without the bilocality assumption, the conditional probabilities for the scenario of Fig. (6) would have the form

$$P(a, b, c|x, y, z) = \sum_{\lambda_1, \lambda_2} \varrho(\lambda_1, \lambda_2) P(a|x, \lambda_1) P(b|y, \lambda_1, \lambda_2) P(c|z, \lambda_2) \quad (31)$$

where $\varrho(\lambda_1, \lambda_2)$ is the probability distribution over the pairs of hidden variables. As usual, $\{x, y, z\}$ and $\{a, b, c\}$ describe, respectively, the inputs and outputs at each local part. Since the two systems received by Bob are being jointly measured, they can be treated as a single entity.

Since ϱ can be chosen such that $\lambda_1 = \lambda_2$ occurs with probability 1, such a model is equivalent to one of the form

$$P(a, b, c|x, y, z) = \sum_{\lambda} \varrho(\lambda) P(a|x, \lambda) P(b|y, \lambda) P(c|z, \lambda), \quad (32)$$

which is the standard description of local realism in a three-party Bell scenario.

The bilocality assumption in addition imposes independence of λ_1 and λ_2 , so that the distribution ϱ is required to factor as $\varrho(\lambda_1, \lambda_2) = \varrho_1(\lambda_1)\varrho_2(\lambda_2)$, which

means that a *bilocal model* is one of the form

$$P(a, b, c|x, y, z) = \sum_{\lambda_1, \lambda_2} \varrho_1(\lambda_1)\varrho_2(\lambda_2) P(a|x, \lambda_1) P(b|y, \lambda_1, \lambda_2) P(c|z, \lambda_2). \quad (33)$$

A direct calculation shows that every such model satisfies, besides the usual no-signaling equations, also the condition

$$\sum_b P(a, b, c|x, y, z) = P(a|x) P(c|z) \quad \forall a, c, x, y, z \quad (34)$$

where $P(a|x)$ and $P(c|z)$ are the marginal behaviors of A and C, respectively.

However, there are no-signaling boxes $P(a, b, c|x, y, z)$ which satisfy (34), but nevertheless are not bilocal, i.e. cannot be written in the form (33). Many examples of this form are even local boxes of the form (32). We will soon see more concrete examples of this in which the “non-bilocality” can be witnessed by an entropic inequality.

Entropic bilocality inequalities. Due to the nonlinearity of the bilocality condition $\varrho(\lambda_1, \lambda_2) = \varrho_1(\lambda_1)\varrho_2(\lambda_2)$, the set of $P(a, b, c|x, y, z)$ of the form (33) is not convex [14], and it is difficult to determine whether a given P lies in this bilocal set or not. This is where entropic inequalities enter: as we will show in the following, they give necessary requirements for P to be bilocal in terms of inequalities *linear* in the entropies of P . This linearity is already visible on the level of the two sources, whose probabilistic independence is equivalent to vanishing mutual information, $I(\lambda_1 : \lambda_2) = 0$, which is a linear entropic equation

$$H(\lambda_1 \lambda_2) = H(\lambda_1) + H(\lambda_2).$$

For the sake of concreteness, we consider the specific scenario where A and C have two available measurement settings to choose from, whereas B always applies the same fixed measurement, so that $x, z \in \{0, 1\}$ and $y = 0$. This corresponds to 5 observables A_0, A_1, B, C_0, C_1 . A subset of these 5 observables is jointly measurable whenever it contains at most one observable of A and at most one of B. Following [6], one can visualize the marginal scenario as in Fig. 7. In addition, the observables A_0 and A_1 are independent of the observables C_0 and C_1 ; therefore, in order to calculate the entropic inequalities for this scenario following the procedure of [6], we need to consider the independence constraint

$$I(A_0 A_1 : C_0 C_1) = 0, \quad (35)$$

which we write out in terms of joint entropies as

$$H(A_0 A_1 C_0 C_1) = H(A_0 A_1) + H(C_0 C_1). \quad (36)$$

Note that the data processing inequality implies that the

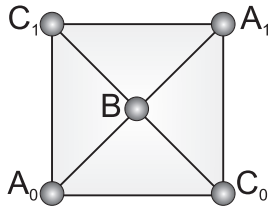


FIG. 7: The simplicial complex of jointly measurable observables in the bilocality scenario discussed in the main text. Two observables are jointly measurable whenever they share an edge; three observables are jointly measurable whenever they are the vertices of one of the four triangles.

independence constraints

$$\begin{aligned} I(A_x : C_0 C_1) &= 0, \\ I(A_0 A_1 : C_z) &= 0, \\ I(A_x : C_z) &= 0, \end{aligned} \quad (37)$$

follow from (35).

We have used the computational approach of [6], augmented by the independence constraints (36) and (37) written out in terms of joint entropies, in order to calculate all the (Shannon-type) entropic inequalities in this bilocality scenario; although including the constraints (37) would not have been strictly necessary, it helps in speeding up the computation. This computation has resulted in 4 equations and 52 tight inequalities. The 4 equations are precisely the independence conditions $I(A_x : C_z) = 0$. The 52 inequalities fall into 10 symmetry classes which we have listed in Table II. The first four inequalities are trivial in the sense that they will hold for any no-signaling box $P(a, b, c|x, y, z)$ satisfying (34), while the other six inequalities can potentially be violated by non-bilocal boxes.

Looking for quantum violations. We now consider the quantum case. Instead of sending out independent hidden variables λ_1 and λ_2 , the two sources now emit entangled quantum states. We take these to be given by generic partially entangled two-qubit states

$$\cos \theta_k |00\rangle + \sin \theta_k e^{i\phi_k} |11\rangle$$

with $k = 1, 2$ indexing the two sources. Then A and C receive one qubit each, while B receives two. Upon choosing the two-qubit measurement of B to be in the Bell basis and numerically optimizing over all projective measurements for A and C , we have not been able to find any quantum violation of any of our 52 entropic bilocality inequalities.

On the other hand, it is not difficult to design some general no-signaling boxes $P(a, b, c|x, z)$ which satisfy (34), but violate some of our entropic inequalities. This applies for example to the family of boxes, for pa-

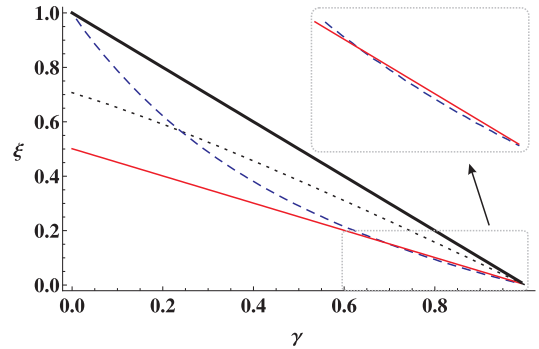


FIG. 8: (Color online) Parameter space of the family (38). The red dot-dashed line delimits the region where (39) and (40), and therefore (38), are local as witnessed by the CHSH inequality. The region above the blue dashed line contains violations of the entropic bilocality inequality #7 from Table II. The black line delimits the region where (39) and (40), and therefore (38), have a quantum realization with a single 3-qubit source. There is a small region (inset) in which our inequality detects that (38) is not bilocal, although the box is tripartite local.

rameters $\xi, \gamma \in [0, 1]$,

$$\begin{aligned} P^{NB}(a, b, c|x, z) \\ = \frac{1}{8} (1 + \xi(-1)^{a \oplus b \oplus c \oplus xz} + (1 - \xi - \gamma)(-1)^{a \oplus b \oplus c}) \end{aligned} \quad (38)$$

This box can be understood as follows. The two outcomes $b \in \{0, 1\}$ both occur with probability $1/2$; if $b = 0$, then this creates between A and C the bipartite box

$$\xi P^{PR} + \gamma P^C + (1 - \xi - \gamma) P^w \quad (39)$$

and if $b = 1$, then the resulting box between A and C is

$$\xi P^{APR} + \gamma P^{Ac} + (1 - \xi - \gamma) P^w, \quad (40)$$

where P^{APR} and P^{Ac} stand for an “anti-PR-box”, defined as in (9) with $C = -1$, and classical anti-correlations, respectively.

Depending on the value of ξ and γ , the box P^{NB} can be tripartite local, tripartite nonlocal but tripartite quantum, or post-quantum tripartite nonlocal. P^{NB} satisfies (34) since the bipartite marginal between A and C is pure white noise, $P^{NB}(a, c|x, z) = P^w(a, c|x, z) = 1/4$.

P^{NB} violates some of our entropic bilocality inequalities. We focus on the inequality 7 from Table II. Fig. 8 shows the region of violations as a function of the parameters ξ and γ . Interestingly, even in the region where P^{NB} is local as a tripartite box, the entropic bilocality inequality can be violated. This witnesses that in those parameter ranges, P^{NB} cannot be written in the form (33), although the box is local and (34) is satisfied.

Entropic Bilocality Inequalities													
#	$H(A_x)$		$H(B)$	$H(C_z)$		$H(A_x B)$		$H(BC_z)$		$H(A_x BC_z)$			
$x \ / \ z \ / \ xz$	0	1	0	0	1	0	1	0	1	00	01	10	11
1	-1	0	-1	0	0	1	0	0	0	0	0	0	0
2	0	0	0	0	0	1	0	0	0	-1	0	0	0
3	1	0	0	1	0	0	0	0	0	-1	0	0	0
4	0	0	1	0	0	-1	0	-1	0	1	0	0	0
5	0	1	0	1	0	1	-1	0	0	-1	0	0	0
6	0	1	0	0	1	1	-1	1	-1	-1	0	0	0
7	1	0	0	1	0	0	0	0	0	0	-1	-1	1
8	1	0	0	1	0	-1	1	0	0	-1	1	0	-1
9	1	0	0	1	0	-1	1	-1	1	1	-1	-1	0
10	0	0	0	0	0	1	0	1	0	-1	-1	-1	1

TABLE II: All classes of entropic bilocality inequalities. We have listed the coefficients of one inequality in each row, and all inequalities are of the form ≤ 0 .

V. CONCLUSION

In this work, we have exploited our general framework [6] for deriving entropic contextuality inequalities and entropic Bell inequalities. The standard methods of computational geometry (like Fourier-Motzkin elimination), which have been applied widely to the computation of tight Bell inequalities, can be used for the computation of tight (Shannon-type) entropic contextuality inequalities (Bell inequalities). Following [17] and related work, our framework also treats nonlocality as a special case of contextuality.

We also had shown in [6] that the family of chained entropic inequalities derived by Braustein and Caves [4] are the only non-trivial facets in the appropriate scenario (Theorem 3). Given that, we have investigated quantum and more general violations of these inequalities both in the Bell scenario case and in the contextuality scenario case. Using a model of joint detection for compatible observables, we noticed that quantum violations of a certain entropic contextuality inequality exist for any positive detection efficiency. Furthermore, we have fully characterized the entropic inequalities for the simplest bilocality scenario. The entropic bilocality inequalities can be violated by general no-signalling correlations respecting the obvious condition (34), but no quantum violations could be found in terms of two independent sources of entangled quantum states.

We have asked the question what the violation of an entropic contextuality inequality or entropic Bell inequality might be a resource for. In the case of the CHSH scenario, we have approached this by noting some superficial similarities with the distillation of nonlocality. We have speculated that the violation of the entropic CHSH inequality may be a sufficient condition for the possibility of distillation. If this would turn out to be true, it may be very useful, since in general it is very difficult to decide which no-signaling boxes are distillable and which

ones are not.

With our general framework, many more possibilities can be explored. For example, the principle of information causality (IC) [28], which has been proposed in order to understand the implausible consequences of super-quantum correlations, is also an entropic inequality. It can be shown that the inequality defining IC is, up to symmetries, the only non-trivial facet of an entropic cone defined by the IC scenario [29]. In principle, modifying the scenario will therefore let us derive many other IC-like principles as entropic inequalities, in particular some with a multipartite flavor.

Since practical computations with entropic cones are extremely demanding, it would be helpful to have more efficient algorithms for the computation of facets of polyhedral cones in order to characterize all (Shannon-type) entropic Bell inequalities for bipartite scenarios with more observables per party, multipartite Bell scenarios, and IC scenarios. To identify nonlinear problems that turn to be linear in terms of entropies, as we have done with bilocality, may also be an appealing line of future research.

Postscript

While finishing this paper, the work [30] has appeared, which also contains some of our results of section II B.

Acknowledgments

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